Analysis 1 - Mid-Sem exam - 2008-09

B.Math (Hons.)

Problem 1. *Show that the ordered field $\mathbb{R}$ of real numbers has the Archimedean property.*

Solution. We need to prove that for every $x > 0, y > 0$, there exists a natural number n such that $nx > y$. Let $\mathbb{N}$ denote the set of natural numbers. We first prove the following lemma.

Lemma: $\mathbb{N}$ is unbounded.

Suppose, $\mathbb{N}$ is bounded. By, l.u.b axiom there exists a least upper bound α such that $n \leq \alpha$ for all $n \in \mathbb{N}$. Now, since α is the least upper bound, $\alpha - 1$ is not an upper bound. Thus, there exists a k such that $k \geq \alpha - 1$. This implies that $k + 2 > \alpha$. Hence we get a contradiction. Thus, $\mathbb{N}$ is unbounded.

Now, given $x > 0$, $y > 0$ consider y/x. By the previous lemma, there exists a $n \in \mathbb{N}$ such that $n > y/x$. Thus we have $nx > y$.

□

Problem 2. *For any two subsets A, B of $\mathbb{R}$, define*

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\}; \\ AB &= \{ab : a \in A, b \in B\} \end{aligned}$$

Show that if A, B are bounded above,

$$sup(A + B) = sup(A) + sup(B).$$

Give an example to show that

$$sup(AB) = sup(A).sup(B),$$

need not be true.

Solution. **(a)** To prove that $sup(A + B) = sup(A) + sup(B)$.

Since, A and B are bounded, $sup(A)$, $sup(B)$, and $sup(A + B)$ are finite.

claim 1: $sup(A + B) \leq sup(A) + sup(B)$.

Let $x \in A + B$ be any arbitrary element. Then $x = a + b$ for some $a \in A$ and $b \in B$. And we know that $a \leq sup(A)$ and $b \leq sup(B)$. Hence $x = a + b \leq sup(A) + sup(B)$.

Hence $x \leq sup(A) + sup(B)$ for all $x \in A + B$. Thus we have $sup(A + B) \leq sup(A) + sup(B)$.

claim 2: $sup(A + B) \geq sup(A) + sup(B)$.

Let $\epsilon > 0$ be arbitrary. Then there exists an $a \in A$ such that $a \geq sup(A) - \epsilon/2$. Similarly, there exists a $b \in B$ such that $b \geq sup(B) - \epsilon/2$. Then $a + b \geq sup(A) + sup(B) - \epsilon$.

But we have $sup(A + B) \geq a + b$. Hence for all $\epsilon > 0$ we have $sup(A + B) \geq sup(A) + sup(B) - \epsilon$. Therefore we have $sup(A + B) \geq sup(A) + sup(B)$.

Thus, from claim 1 and claim 2, we have $sup(A + B) = sup(A) + sup(B)$.

(b) To show that $sup(AB) = sup(A)sup(B)$ is not necessarily true.

Let $A = [-1, 0]$ and $B = [-1, 0]$. We have $sup(A) = sup(B) = 0$, and hence $sup(A).sup(B) = 0$.

Now, we have that $1 \in A.B$. Thus $sup(A, B) \geq 1$. Hence we cannot have $sup(AB) = sup(A)sup(B)$.

□

Problem 3. *Show that intervals* $[0, 1]$ *and* $(0, 1)$ *have same cardinality.*

Solution. Let

$$H = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

Define the function $f : (0, 1) \to [0, 1]$ such that

$$f(x) = \begin{cases} 0 & if\ x = 1/2 \\ 1 & if\ x = 1/3 \\ 1/(n-2) & if\ x = 1/n\ ; n \geq 3\ and\ x \in \mathbb{N} \\ x & if\ x \notin H \end{cases}$$

Now, define $g : [0, 1] \to (0, 1)$ as

$$f(x) = \begin{cases} 1/2 & if\ x = 0 \\ 1/3 & if\ x = 1 \\ 1/(n+2) & if\ x = 1/n\ ; n \geq 3\ and\ x \in \mathbb{N} \\ x & if\ x \notin H \cup \{0, 1\} \end{cases}$$

Now note that $f \circ g = I$ and $g \circ f = I$. Hence we have that f is a bijection, and the intervals $[0, 1]$ and $(0, 1)$ have same cardinality. □

Problem 4. *Let* $\{a_n\}$ *be a sequence of real numbers converging to* $a \in \mathbb{R}$. *Define*

$$b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

for $n \geq 1$. *Show that* $\{b_n\}$ *is a converging sequence converging to* a.

Solution. Let $\epsilon > 0$ be arbitrary. Since $\{a_i\}$ is converging sequence, we can choose a K such that

$$| a_i - a | < \frac{\epsilon}{2} \qquad \forall i \geq K$$

Let

$$M = \sum_{i=1}^{K} | a_i - a |$$

For $n \geq K$, we have

$$\begin{aligned} |\ b_n - a\ | &= \frac{1}{n}\left|\sum_{i=1}^{n} a_i - na\right| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} |\ a_i - a\ | \\ &= \frac{1}{n}\sum_{i=1}^{K} |\ a_i - a\ | + \frac{1}{n}\sum_{i=K+1}^{n} |\ a_i - a\ | \\ &\leq \frac{M}{n} + \frac{\epsilon}{2}\frac{n-K}{n} \\ &\leq \frac{M}{n} + \frac{\epsilon}{2} \end{aligned}$$

Now there exists a N such that $M/n < \epsilon/2$ for all $n \geq N$. Therefore for all $n \geq max\{N.K\}$, we have

$$|\ b_n - a\ | \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.$$

Since ϵ is arbitrary b_n converges to a. $\square$

Problem 5. *Show that every Cauchy sequence of real numbers is convergent.*

Solution. Let $\{x_n\}$ be a cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$ be arbitrary. There exist $N > 0$, such that for all $m, n \geq N$, we have that $|x_n - x_m| < \epsilon/2$. In particular we have $|x_n - x_N| < \epsilon/2$. Or equivalently,

$$x_n \in (x_N - \epsilon/2, x_N + \epsilon/2) \qquad \forall n \geq N \tag{1}$$

From this we make the following observations:

(a) For all $n > N$ we have $x_n > x_N - \epsilon/2$

(b) If $x_n > x_N + \epsilon/2$, then $n \in \{1, 2, \cdots N-1\}$. Thus, the set of n such that $x_n \geq x_N + \epsilon/2$ is finite.

Let $S = \{x \in \mathbb{R} :$ there exists infintely many n such that$x_n \geq x\}$. We claim that S is non-empty, bounded above and that $sup(S)$ is the limit of the given sequence.

Let $\epsilon = 1$. There there exists a N such that (1) holds. Therefore $x_N - 1/2$ belongs to S. Hence S is non empty.

We claim that $x_N + 1$ is an upper bound for S. If this were not true, there exists a $y \in S$ such that $y > x_N + 1$ and $x_n \geq y$ for infinitely many n. This implies that $x_n > x_N + 1$ for infinitely many n. This results in a contradiction from (b). Hence $x_N + 1$ is an upper bound for S.

By the LUB axiom, there exists a $l \in \mathbb{R}$, such that $sup(S) = l$. We claim that $\lim x_n = l$. Let $\epsilon > 0$ be given. As l is an upper bound for S and $x_N - \epsilon/2 \in S$, from above, we infer that $x_N - \epsilon/2 \leq l$. Since l is the least upperbound for S and $x_N + \epsilon/2$ is an upper bound for S, we see that $l \leq x_N + \epsilon/2$. Thus we have $x_N - \epsilon/2 \leq l \leq x_N + \epsilon/2$ or

$$|x_N - l| \leq \epsilon/2$$

For $n \geq N$, we have

$$\begin{aligned} |x_n - l| &\leq |x_n - x_N| + |x_N - l| \\ &\leq \epsilon/2 + \epsilon/2 \end{aligned}$$

Thus we have shown that $\lim x_n = l$.

$\square$

Problem 6. *Let A be a subset of $\mathbb{R}$. Show that the interior of A is the largest open set contained in A.*

Solution. Denote interior of A as $int(A)$. To show that, $int(A)$ is the largest open set contained in A.

Firstly, by definition of $int(A)$, we have that $int(A)$ is an open set contained in A. Let X be the largest open set. We have that $X \supseteq int(A)$.

Let $x \in X$. Since X is open, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset X \subset A$. From the definition of $int(A)$ we have $X \subseteq int(A)$.

Hence $X = int(A)$.

□

Problem 7. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define a new function $g_+ : \mathbb{R} \to \mathbb{R}$ by*

$$g_+(x) = \begin{cases} 0 & if\ g(x) < 0, \\ x & otherwise. \end{cases}$$

Show that if g is continuous then g_+ is continuous. Give an example where g_+ is continuous but g is not continuous.

Solution. **(a)** To show that g_+ is continuous. Firstly, note that

$$g_+(x) = \frac{g(x)}{2} + \frac{\mid g(x) \mid}{2}$$

claim 1 : Let $x \in \mathbb{R}$. If g is continuous at x then $\mid g \mid$ is continuous at x.

Let $\epsilon > 0$ be arbitrary. Since, g is continuous at x there exists a $\delta > 0$ such that if $\mid x - y \mid < \delta$, then

$$\mid g(x) - g(y) \mid < \epsilon$$

Now,

$$\Big| \mid g(x) \mid - \mid g(y) \mid \Big| \leq \mid g(x) - g(y) \mid \leq \epsilon$$

thus we have that at x, there exists a $\delta > 0$ such that if $\mid x - y \mid < \delta$, then

$$\Big| \mid g(x) \mid - \mid g(y) \mid \Big| < \epsilon$$

Thus, $\mid g \mid$ is continuous.

Since, linear combination of continuous functions is continuous, we have g_+ is continuous.

(b) To show that if g_+ is continuous, g need not be continuous.

Let $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(x) = \begin{cases} 0 & if\ x \in \mathbb{Q} \\ -1 & if\ x \in \mathbb{Q}^c \end{cases}$$

Note that g is discontinuous. Now, we have $g_+(x) = 0$ for all $x \in \mathbb{R}$, which is continuous. Hence if g_+ is continuous, g need not be continuous. □